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On a Lagrangian for non-minimally coupled gravitational and electromagnetic fields

H A Buchdahl

Department of Theoretical Physics, Faculty of Science, Australian National University, Canberra ACT 2600, Australia

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Abstract. An arbitrarily chosen Lagrangian L for non-minimally coupled gravitational and electromagnetic fields will usually lead to higher-order field equations, in the sense that the functional derivatives of L with respect to the gravitational potential g_{ij} and the electromagnetic potential ϕ_i will involve at least the third, instead of merely the second, derivatives of these quantities. By temporarily contemplating a five-dimensional formalism this paper uncovers an exceptional case in which one is led to second-order equations. The result obtained is in agreement with the conclusions reached by Horndeski by quite different means.

1. Introduction

Einstein's theory of gravitation is governed by a variational principle, the Lagrangian of which is, in the absence of a cosmological term,

$$L = R + \kappa M \tag{1.1}$$

where R is the scalar curvature of a four-dimensional Riemann space V_4 (here of signature -2), M is the matter Lagrangian and κ is a constant. Moreover, by prescription M is to be such that the energy–momentum tensor T^{kl} which it implies coincides with that object which arises from the replacement of the Minkowski metric η_{kl} by the metric g_{kl} of V_4 and of partial by covariant derivatives in the special relativistic energy–momentum tensor describing whatever stress-energy–momentum may be present. It is just this prescription which characterises what is usually known as the principle of minimal gravitational coupling. Theories have occasionally been considered in which this principle is abandoned and then M contains the Riemann tensor (and possibly its covariant derivatives) explicitly. This has been done, for instance by Goenner (1976). As a result of this work Ehrenpreis (1977) later examined the case of a non-minimally coupled electromagnetic field in great detail. Again, Prasanna (1971) suggested the addition of a term

$$M' := \text{constant} \times R^{ijkl} f_{ij} f_{kl} \tag{1.2}$$

to the Einstein–Maxwell Lagrangian, where f_{ij} is the electromagnetic field tensor.

It is precisely one particular aspect of the addition of non-minimally coupled electromagnetic source terms to the Einstein–Maxwell Lagrangian which I wish to consider here from a formal point of view. It is this: the Einstein–Maxwell equations are of the second differential order in the sense that the functional derivatives of L with respect to g_{ij} on the one hand and the components ϕ_k of the electromagnetic potential on the other all involve at most the second derivatives of these quantities. However, as soon as one interferes with the principle of minimal gravitational coupling the functional derivatives just referred to will usually involve the derivatives of at least the third order of the g_{ij} and ϕ_k, i.e. one arrives at higher-order differential equations. This is the case, for instance, in the context of (1.2). The question naturally arises whether it is possible to choose non-minimally coupled electromagnetic terms in L so that the resulting field equations are still of the second order.

The answer to the question just posed is in the affirmative: if (1.1) is a minimally coupled Lagrangian—in which M is therefore an invariant of f_{kl}—then one can add to L a certain further invariant I such that the resulting minimally coupled Lagrangian still generates second-order field equations. This result is known from the work of Horndeski (1976), who finds I explicitly in a somewhat different context and effectively demonstrates its uniqueness. However, the derivation of his result requires work of such length that the paper just quoted, long as it is, in fact gives only an outline of a proof of his main theorem. Here I therefore approach the problem of constructing I by a method, interesting in its own right, which rests upon the consideration of a quadratic invariant of the Riemann tensor whose functional derivative happens to vanish in a V_4. A temporary excursion to a well known five-dimensional formalism leads directly to an explicit expression for I. Its functional derivatives with respect to ϕ_k and to g_{kl} are also obtained explicitly. Their correctness may be checked by making use of the two differential identities which they must satisfy.

The procedure adopted here strongly suggests, but does not prove, that I is unique. That it is in fact so is shown by the work of Horndeski already quoted.

2. Five-dimensional Lagrangians

In a V_4 there are five independent quadratic invariants of the Riemann tensor:

$$K_1 := R^2 \qquad K_2 := R_{ij}R^{ij} \qquad K_3 := R_{ijkl}R^{ijkl} \qquad K_4 := \mathrm{e}^{klmn}R_{klpq}R_{mn}{}^{pq}. \tag{2.1}$$

It is known that $K := K_1 - 4K_2 + K_3$ and K_4 are functionally constant, i.e. their functional derivatives vanish identically, as is perhaps shown most easily by appealing to the calculus of two-spinors (Buchdahl 1960). Now K is defined in a V_n also when $n > 4$, and then it has the special property that $\delta K/\delta g_{ij}$ does not involve covariant derivatives of the Riemann tensor, i.e. it is a *second*-order differential concomitant of g_{kl}. This can be shown directly by first rewriting K in a different form (see § 3) or by observing that covariant derivatives of R_{ijkl} appear in the functional derivatives of K_1, K_2 and K_3 only in the combinations $-2R^{;ij} + 2g^{ij}\Box R$, $\Box R^{ij} - R^{;ij} + \frac{1}{2}g^{ij}\Box R$ and $4\Box R^{ij} - 2R^{;ij}$, respectively (Buchdahl 1948).

In the present context the case $n = 5$ is of particular interest. Accordingly, let greek and italic indices henceforth go over the ranges 1 to 5 and 1 to 4 respectively. At the same time, the metric tensor of the V_5 and its algebraic and differential concomitants will be distinguished by bars. For example,

$$\bar{K} = \bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} - 4\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \bar{R}^2. \tag{2.2}$$

The metric of the V_5 shall now have the particular form

$$\overline{ds}^2 = \bar{g}_{\mu\nu}\, dx^\mu\, dx^\nu =: g_{ij}\, dx^i\, dx^j + (dx^5 + \phi_i\, dx^i)^2 \tag{2.3}$$

where g_{ij} and ϕ_k are to depend upon the x^l alone. Concomitantly the allowed group of transformations is to consist of the elements

$$x^{i'} = f^{i'}(x^1, x^2, x^3, x^4) \qquad x^{5'} = x^5 + \theta(x^1, x^2, x^3, x^4). \tag{2.4}$$

The specification (2.3) and (2.4) simply amounts to the transition to a five-dimensional view of the gravitational and electromagnetic fields, g_{ij} being the metric tensor of the 'physical' space–time V_4 and ϕ_i (a constant factor aside), the electromagnetic potential. In other words, for purely formal purposes the standpoint of the familiar Klein–Kaluza theory (Pauli 1958) is temporarily being adopted except to the extent that Lagrangians more general than $\bar{R}$ are now contemplated. In particular, choose

$$\bar{L} = \bar{R} + \alpha\bar{K} \tag{2.5}$$

where α is a constant. In four dimensions $\bar{R}$ reproduces the Einstein–Maxwell Lagrangian, and it remains to examine $\bar{K}$.

3. The invariant I

The reduction of $\bar{K}$ to four-dimensional form in the first place requires one to write down the components of $\bar{R}^\mu{}_{\nu\rho\sigma}$ and its concomitants in terms of $R^l{}_{ijk}$, f_{mn}, $f_{pq;s}$ and their concomitants. This task may be greatly reduced in scope by observing that $\bar{K}$ is gauge invariant, i.e. it cannot contain the ϕ_j explicitly. Accordingly, it suffices to omit from the $\bar{R}^\mu{}_{\nu\rho\sigma}$ all terms which do depend upon ϕ_j explicitly. Replacing the equality sign by the sign $\stackrel{*}{=}$ in any relation which is valid when terms of the kind in question are rejected, one finds that

$$\begin{aligned} \bar{R}^l{}_{ijk} &\stackrel{*}{=} R^l{}_{ijk} + \tfrac{1}{2}(f_{i[k}f_{j]}{}^l + f_{jk}f_i{}^l) \\ \bar{R}^5{}_{ijk} &\stackrel{*}{=} \tfrac{1}{2}f_{jk;i} \\ \bar{R}^5{}_{ij5} &\stackrel{*}{=} -\tfrac{1}{4}f_{im}f_j{}^m, \end{aligned} \tag{3.1}$$

from which the various concomitants of $\bar{R}^\mu{}_{\nu\rho\sigma}$ follow easily, bearing in mind that

$$\bar{g}^{ij} = g^{ij} \qquad \bar{g}^{i5} \stackrel{*}{=} 0 \qquad \bar{g}^{55} \stackrel{*}{=} 1. \tag{3.2}$$

Then

$$\bar{K}_3 \stackrel{*}{=} \bar{R}_{ijkl}\bar{R}^{ijkl} + 4\bar{R}^5{}_{lmn}\bar{R}^{5lmn} + 4\bar{R}^5{}_{lm5}\bar{R}^{5lm5}. \tag{3.3}$$

Inserting (3.1) in this, $\bar{K}_3$ will contain certain terms which constitute an invariant of f_{ij} alone. Such terms need not be retained explicitly since they represent minimally coupled terms in L, i.e. they may be thought of as absorbed in M. Any invariant from which such terms are omitted will be distinguished by an asterisk. Thus (3.3) gives

$$\bar{K}_3{}^* = K_3 - \tfrac{3}{2}R_{ijkl}f^{ij}f^{kl} + f^{ij;k}f_{ij;k}. \tag{3.4}$$

With the abbreviations

$$s^i := f^{ij}{}_{;j} \qquad t_{ij} := f_{ik}f_j{}^k \qquad t := t^i{}_i \tag{3.5}$$

one finds likewise that

$$\bar{K}_2{}^* = K_3 + R_{ij}t^{ij} + \tfrac{1}{2}s_is^i \qquad \bar{K}_1{}^* = K_1 + \tfrac{1}{2}Rt. \tag{3.6}$$

Hence

$$\bar{K}^* = K - \tfrac{3}{2}R_{ijkl}f^{ij}f^{kl} - 4t^{ij}R_{ij} + \tfrac{1}{2}Rt + (f^{ij;k}f_{ij;k} - 2s_is^i). \tag{3.7}$$

The appearance of the derivative terms in this must be illusory. In fact, since $\bar{K}^*$ is to be part of a Lagrangian any divergence may be rejected from it. Thus the last two terms of (3.7) may be replaced by $B := (2f_i{}^j{}_{;jk} - f_{ik;}{}^j{}_j)f^{ik}$, having rejected the divergence $(f^{ij;k}f_{ij} - 2f_i{}^j{}_{;j}f^{ik})_{;k}$. Now

$$B = f^{ik}[(2f_{ij;k} - f_{ik;j})^{;j} - 2R_{mikj}f^{mj} + 2R_{mk}f_i{}^m].$$

The divergence which appears here vanishes because $f_{[ij;k]} = 0$ and one is left with

$$B = R_{ijkl}f^{ij}f^{kl} + 2R_{ij}t^{ij},$$

use having been made of the identities $R_{i[jkl]} = 0$, $f_{(ij)} = 0$ to rewrite the first term on the right.

Returning to (3.7), K may also be omitted since its functional derivative vanishes. Thus there finally remains the invariant

$$I := -\tfrac{1}{2}f^{ij}f^{kl}R_{ijkl} - 2f^{ik}f^j{}_kR_{ij} + \tfrac{1}{2}f^{ij}f_{ij}R, \tag{3.8}$$

which must be of the required kind (see also equation (4.3)).

4. Alternative form of $\bar{K}$

As has been seen, $\bar{K}$ and I differ from each other only by irrelevant terms, i.e. (i) those which have the form of a divergence and (ii) those which constitute a minimally coupled invariant of f_{ij}. By writing $\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}$ as a determinant of simple Kronecker deltas, it may be confirmed directly that

$$\bar{K} = \tfrac{1}{4}\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}R_{\mu\nu}{}^{\alpha\beta}R_{\rho\sigma}{}^{\gamma\delta}. \tag{4.1}$$

The calculation of the functional derivative $\delta\bar{K}/\delta\bar{g}_{\mu\nu}$ is straightforward. It is not necessary to go through the details: it suffices to observe that after one integration by parts there appears in the variation of $\int \bar{K}(-\bar{g})^{1/2}\,dx$ a term

$$\int (-\bar{g})^{1/2}\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}\bar{g}^{\delta\tau}\bar{R}^{\alpha\beta}{}_{\mu\nu;\rho}\delta\bar{\Gamma}^{\gamma}{}_{\tau\sigma}\,dx$$

which vanishes because of the identity of Bianchi. It is in this way that higher derivatives disappear from $\delta\bar{K}/\delta\bar{g}_{\mu\nu}$. Explicitly

$$\delta\bar{K}/\delta g_{\mu\nu} = \tfrac{1}{2}(\bar{g}^{\mu\nu}\bar{K} + \delta^{\lambda\omega\rho\sigma}_{\alpha\beta\gamma\delta}\bar{g}^{\delta(\mu}\bar{R}^{\nu)\gamma}{}_{\rho\sigma}\bar{R}_{\lambda\omega}{}^{\alpha\beta}), \tag{4.2}$$

which is obviously of the second differential order. For purposes of explicit calculation (4.2) does not appear to be useful.

It may be noted in passing that (3.8) may be written as

$$I = -\tfrac{1}{8}\delta^{abcd}_{klmn}R_{ab}{}^{kl}f_{cd}f^{mn}. \tag{4.3}$$

5. The functional derivative $\delta I/\delta\phi_l$

The functional derivative $P^l := \delta I/\delta\phi_l$ can be derived in the usual way and no further comment is required. In a few lines one arrives at the result

$$P^l = 2R^{ijkl}f_{ij;k} - 4R_{ij}f^{li;j} - 4R^{il}s_i + 2Rs^l. \tag{5.1}$$

6. The functional derivative $\delta I/\delta g_{kl}$

In contrast to the calculation of P^i that of $P^{ij} := \delta I/\delta g_{ij}$ is a rather lengthy process. Upon varying $\int I(-g)^{1/2}\,\mathrm{d}x$ with respect to g_{ij} and integrating by parts where required, one arrives at first at an integrand which consists additively of 18 terms. Of these, seven involve second derivatives of f_{st}. Evidently it must be possible to rewrite them so that all second derivatives disappear.

Of the seven terms just referred to, only two have g^{ij} as a factor. They are

$$g^{ij}(\tfrac{1}{2}\Box t - t^{kl}{}_{;kl}) =: g^{ij}D.$$

Let ... indicate any terms free of second derivatives of f_{st}. Then

$$\begin{aligned} D &= f^{st}\Box f_{st} - f^{km}{}_{;kl}f^l{}_m - f^{km}f^l{}_{m;kl} + \dots \\ &= f^{st}(f_{st;}{}^m{}_m - f^m{}_{t;ms} - f^m{}_{t;sm}) + \dots . \end{aligned}$$

Interchange of the order of covariant differentiation merely contributes terms to .... Thus

$$\begin{aligned} D &= f^{st}(f_{st;m} + 2f_{ms;t})^{;m} + \dots \\ &= f^{st}(f_{st;m} + f_{ms;t} + f_{tm;s}) + \dots \\ &= \dots . \end{aligned}$$

The other five terms of the seven referred to above may be dealt with in the same way, even if the details are a little cumbersome. Explicitly, collecting all the terms hitherto represented merely by ... one arrives at the result that

$$\begin{aligned} P^{ij} = {} & -6f_{st}f^{(i}{}_mR^{j)stm} + 2t_{mn}R^{imnj} + 3f^{is}f^{jt}R_{st} + 4t^{m(i}R^{j)}{}_m - \tfrac{1}{2}tR^{ij} - t^{ij}R + s^is^j + 2f^{m(i;j)}s_m \\ & - f^i{}_{s;t}f^{jt;s} - f^{st;i}f_{st;}{}^j + g^{ij}(-\tfrac{3}{4}f^{st}f^{mn}R_{mnst} - 2t^{st}R_{st} + \tfrac{1}{4}tR + \tfrac{1}{2}f_{st;m}f^{st;m} - s^ms_m). \end{aligned} \tag{6.1}$$

7. Differential identities

Consider the variation of $\int L(-g)^{1/2}\,\mathrm{d}x$, where L is any gauge-invariant Lagrangian, i.e. L is a function of the g_{ij} and f_{kl} only. Thus

$$\Delta := \delta\int L(g_{ij}, f_{kl})(-g)^{1/2}\,\mathrm{d}x = \int (P^{ij}\delta g_{ij} + P^i\delta\phi_i)(-g)^{1/2}\,\mathrm{d}x.$$

Suppose the variations to be due to an infinitesimal coordinate transformation which vanishes on the boundary. Then

$$\Delta := \int [2P^{ij}\xi_{i;j} + P^i(\phi_{i;j}\xi^j + \phi_j\xi^j{}_{;i})](-g)^{1/2}\,\mathrm{d}x,$$

where ξ^i is an arbitrary infinitesimal vector field which vanishes on the boundary. Integration by parts leads to the conclusion that

$$\int (2P_i^j{}_{;j} - f_{ij}P^j + P^j{}_{;j}\phi_i)\xi^i(-g)^{1/2}\,\mathrm{d}x = 0.$$

Since ξ^i is arbitrary within the region of integration and since L is gauge invariant it follows that

$$P^{kl}{}_{;l} = \tfrac{1}{2}f^{kl}P_l \qquad P^l{}_{;l} = 0. \tag{7.1}$$

The 'total current' $s^i + \text{constant} \times P^i$ is thus still conserved.

The identities (7.1) evidently provide a reliable check upon equations (5.1) and (6.1). The required calculations are, however, even lengthier than those which led to (6.1). For instance, upon forming the divergence of (6.1) one is at first confronted with an expression which consists additively of 41 distinct terms. Of these, 12 cancel mutually by inspection as they stand. The remaining 29 fall fairly naturally into several groups which can be reduced one by one. The terms which arise from this reduction can again be grouped together, to be further reduced. At any rate, the final outcome of all this is that (5.1) and (6.1) indeed satisfy (7.1), as required.

8. Concluding remark

If (in a V_4) $P_{ijkl} := R_{ijst}R_{kl}{}^{st}$, the invariant $K_4 = e^{ijkl}P_{ijkl}$ is also functionally constant. However, K_4, unlike K, is not defined in a V_n $(n \neq 4)$. Any procedure of the kind adopted above would therefore hinge on finding an appropriate generalisation of K_4 to a V_5. The only possibility seems to be this: in place of K_4 contemplate $(K_4{}^2)^{1/2}$ and then generalise the latter to a V_5, i.e. contemplate the invariant

$$J := (\eta\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}P^{\alpha\beta\gamma\delta}P_{\mu\nu\rho\sigma})^{1/2} \tag{8.1}$$

where $\eta = \operatorname{sgn} g$. Then it turns out that covariant derivatives of $R_{\mu\nu\rho\sigma}$ appear in $\delta J/\delta g_{\xi\eta}$ only in the term

$$4\eta\delta^{\mu\nu\rho(\xi}_{\alpha\beta\gamma\delta}[R^{\eta)\tau}{}_{\mu\nu}(J^{-1}P^{\alpha\beta\sigma\delta})_{;\rho}]_{;\tau} \tag{8.2}$$

and there is no reason why this should vanish. Only in a V_4 can one write the Kronecker delta as the product of two e tensors so that the expression corresponding to (8.2) becomes

$$4\epsilon^{klm(2}[R^{t)n}{}_{kl}(J^{-1}e_{abcd}P^{abcd})_{;m}]_{;n}$$

and this vanishes because $J^{-1}e_{abcd}P^{abcd} = 1$. I therefore gives every impression of being unique, in harmony with the result of Horndeski (1976).

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